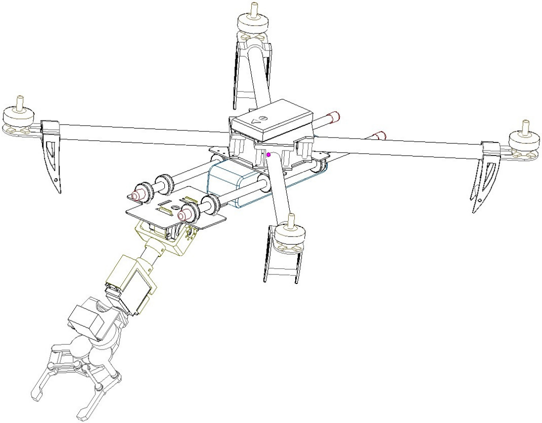
# A Dual Quaternion Chattering-Free Sliding-Mode Controller for a Quadrotor Aerial Manipulator

**Hernán ABAUNZA,**
**Pedro CASTILLO**

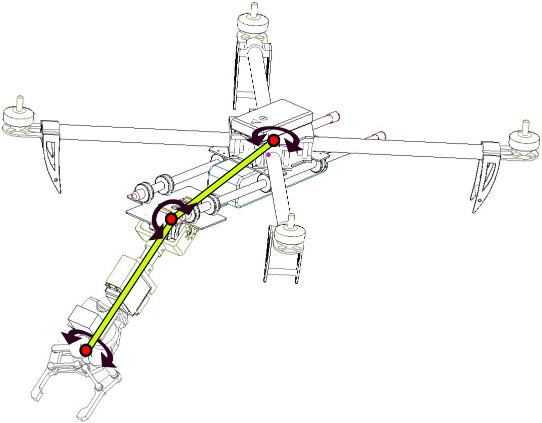
Sorbonne Universités, Université de Technologie de Compiegne,
CNRS, UMR 7253 Heudiasyc, Compiégne France.
E-mail: (habaunza, castillo)@hds.utc.fr

# Contents

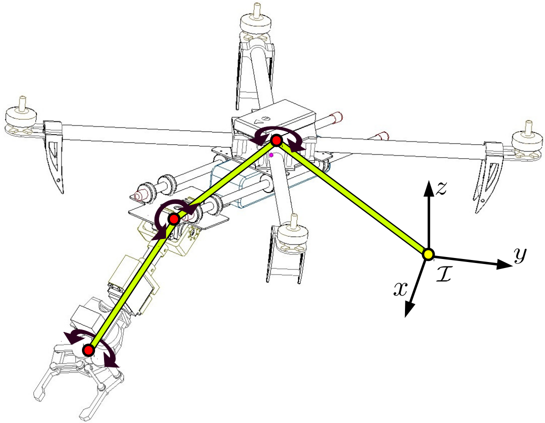
# Introduction
## Quadrotor Aerial Manipulator



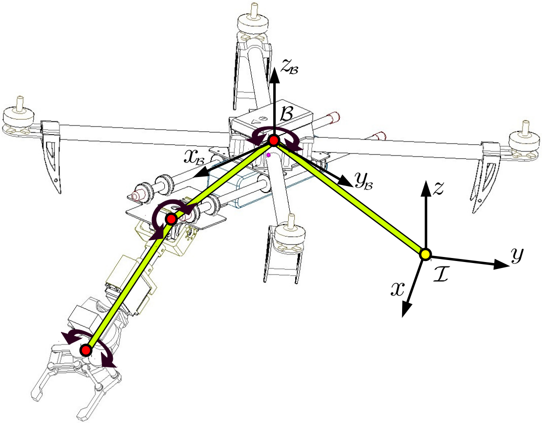
# Introduction
## Quadrotor Aerial Manipulator



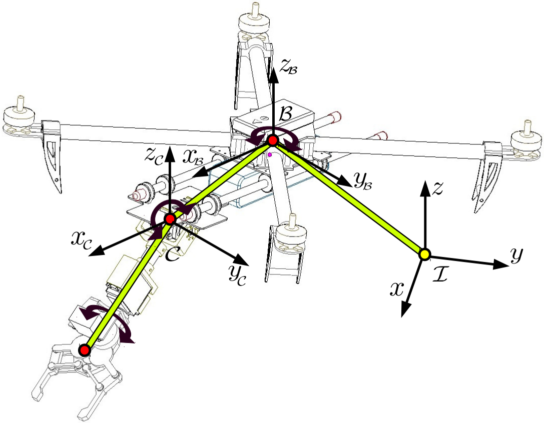
# Introduction
## Quadrotor Aerial Manipulator



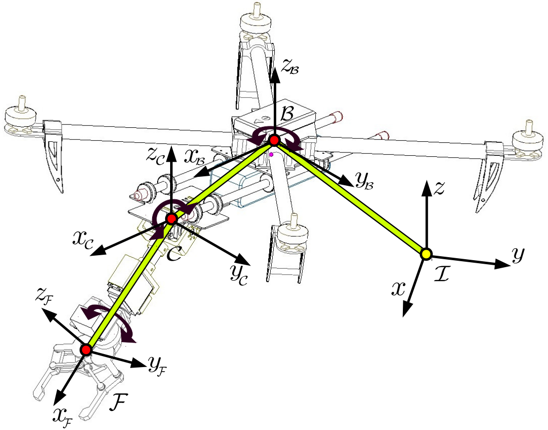
# Introduction
## Quadrotor Aerial Manipulator



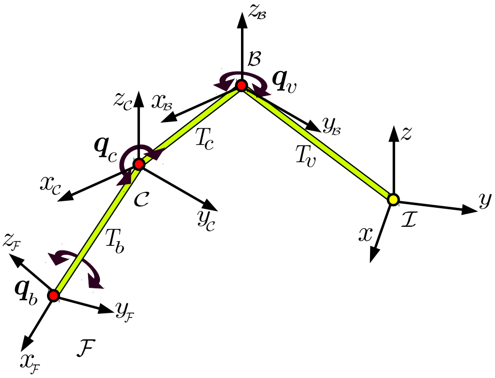
# Introduction
## Quadrotor Aerial Manipulator



# Introduction
## Quadrotor Aerial Manipulator



# Introduction
## Quadrotor Aerial Manipulator



# Introduction

## Dual Quaternion Kinematics

Dual quaternions are a kind of dual numbers, composed by quaternions as

$$\hat{q} = \boldsymbol{q}_r + \boldsymbol{q}_d \epsilon, \;\; \epsilon \neq 0, \;\; \epsilon^2 = 0, \tag{1}$$

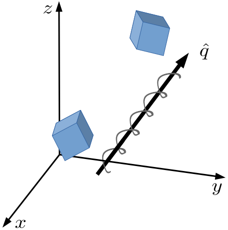
where $\boldsymbol{q}_r, \boldsymbol{q}_d \in \mathbb{H}$ are known as the real and dual parts of $\hat{q}$ respectively.

# Introduction
## Dual Quaternion Kinematics

Supposing a rigid body is subjected to a rotation represented by a quaternion $\boldsymbol{q} = \cos\left(\frac{\vartheta}{2}\right) + \vec{n}\sin\left(\frac{\vartheta}{2}\right)$, $\vartheta \in \mathbb{R}$, $\vec{n} \in \mathbb{R}^3$, followed by a translation $T \in \mathbb{R}^3$ , then its dual quaternion transformation can be expressed as

$$\hat{q} = \boldsymbol{q} + \frac{1}{2}\boldsymbol{q} \otimes T\epsilon, \tag{2}$$

where $\otimes$ is a quaternion product, and $T$ can be considered as a quaternion which real part equals zero.



# Introduction
## Dual Quaternion Kinematics

The dual quaternion logarithmic mapping is defined as

$$\ln \hat{q} = \ln \boldsymbol{q} + \frac{1}{2} T\epsilon = \frac{1}{2} \vec{n}\vartheta + \frac{1}{2} T\epsilon, \tag{3}$$

The derivative of a unit dual quaternion is given bys

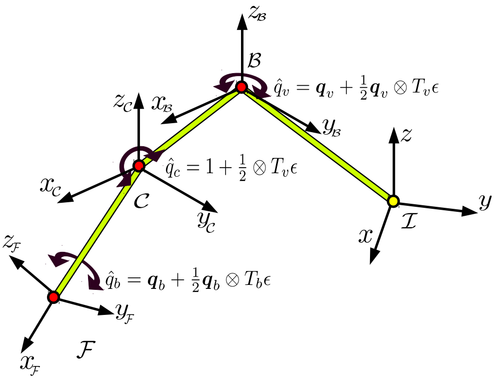
$$\dot{\hat{q}} = \frac{1}{2} \hat{q} \otimes \hat{\xi} \tag{4}$$

where

$$\hat{\xi} = \omega + [\omega \times T + \dot{T}]\epsilon, \tag{5}$$

is known as the *twist* (combination of rotational and translational velocities).

# Aerial manipulator dual quaternion modeling




$$\hat{q}_f = \hat{q}_v \otimes \hat{q}_c \otimes \hat{q}_b, \qquad (6)$$

# Aerial manipulator dual quaternion modeling

## Rigid body Modeling

According to Newton's equations of motion:

$$J\dot{\omega} + \omega \times (J\omega) = \tau \quad , \qquad m\ddot{T} = F,$$

then, considering $\dot{\hat{\xi}} = \dot{\omega} + [\dot{\omega} \times T + \omega \times \dot{T} + \ddot{T}]\epsilon$, it yields

$$\frac{d}{dt}\begin{bmatrix} \hat{q} \\ \hat{\xi} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\hat{q} \otimes \hat{\xi} \\ \\ -J^{-1}(\omega \times J\omega + \tau) + \Big[J^{-1}(\tau \times T - (\omega \times J\omega) \times T) \\ +\omega \times \dot{T} + m^{-1}F\Big]\epsilon \end{bmatrix} \qquad (7)$$

# Aerial manipulator dual quaternion modeling
## Quadrotor subsystem

Considering the quadrotor as a symmetrical rigid body which rotates around its center of mass, its dual quaternion dynamic model is then given by

$$\begin{bmatrix} \dot{\hat{q}}_v \\ \dot{\hat{\xi}}_v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\hat{q}_v \otimes \hat{\xi}_v \\ \hat{F}_v + \hat{u}_v \end{bmatrix} \tag{8}$$

Where control and external forces are included in terms $\hat{F}_v$ and $\hat{u}_v$,

$$\begin{aligned} \hat{F}_v &= -J_v^{-1}(\omega_v \times J_v \omega_v) \\ \hat{u}_v &= J_v^{-1}\tau_v + [m_v^{-1}F_v]\epsilon \\ &= J_v^{-1}(\tau_u + \tau_{arm}) + [m_v^{-1}\boldsymbol{q} \otimes F_{th} \otimes \boldsymbol{q}^* + \vec{g}_v]\epsilon, \end{aligned},$$

where $\omega_v$ is the vehicle's angular velocity and $T_v$ defines its position with respect to its local frame, $J_v$ denotes the quadrotor's inertia matrix, $m_v$ represents its mass, $\tau_v, F_v, F_{th} \in \mathbb{R}^3$ correspond to the total torque, total force, and vertical quadrotor thrust vectors respectively.

# Aerial manipulator dual quaternion modeling
## Robotic arm subsystem

the total transformation of the arm's end-effector can be computed as

$$\hat{q}_f = \hat{q}_v \otimes \hat{q}_c \otimes \hat{q}_b, \tag{9}$$

where $\hat{q}_v$ represents the quadrotor's transformation, $\hat{q}_c$ is a constant translation from the UAV's center of mass to the first joint of the robotic arm, and $\hat{q}_b$ denotes the limb's kinematics, such that

$$\hat{q}_f = \boldsymbol{q}_v \otimes \boldsymbol{q}_b + \frac{1}{2}\left[\boldsymbol{q}_v \otimes \boldsymbol{q}_b \otimes T_b + \boldsymbol{q}_v \otimes (T_c + T_v) \otimes \boldsymbol{q}_b\right]\epsilon. \tag{10}$$

# Aerial manipulator dual quaternion modeling

## Robotic arm subsystem

Differentiating (10) results in

$$\begin{aligned}\dot{\hat{q}}_f = \quad & \dot{\boldsymbol{q}}_v \otimes \boldsymbol{q}_b + \boldsymbol{q}_v \otimes \dot{\boldsymbol{q}}_b + \tfrac{1}{2}[\dot{\boldsymbol{q}}_v \otimes \boldsymbol{q}_b \otimes T_b + \boldsymbol{q}_v \otimes \dot{\boldsymbol{q}}_b \otimes T_b + \boldsymbol{q}_v \otimes \boldsymbol{q}_b \otimes \dot{T}_b \\ & + \dot{\boldsymbol{q}}_v \otimes (T_c + T_v) \otimes \boldsymbol{q}_b + \boldsymbol{q}_v \otimes \dot{T}_v \otimes \boldsymbol{q}_b + \boldsymbol{q}_v \otimes (T_c + T_v) \otimes \dot{\boldsymbol{q}}_b]\epsilon\end{aligned}, \tag{11}$$

Considering $\dot{\hat{q}}_f = \frac{1}{2}\hat{q}_f \otimes \hat{\xi}_f \Rightarrow \hat{\xi}_f = 2\hat{q}_f^* \otimes \dot{\hat{q}}_f$, (10) and (11) can be combined to obtain the total twist of the end effector.

$$\begin{aligned}\hat{\xi}_f = \quad & \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b + \omega_b + \tfrac{1}{2}[-T_b \times \omega_b - T_b \times \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b \\ & - \boldsymbol{q}_b^* \otimes (T_c + T_v) \times \omega_v \otimes \boldsymbol{q}_b + \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b \times T_b + \omega_b \times T_b \\ & + \dot{T}_b + \boldsymbol{q}_b^* \otimes \omega_v \times (T_c + T_v) \otimes \boldsymbol{q}_b + 2\boldsymbol{q}_b^* \otimes \dot{T}_v \otimes \boldsymbol{q}_b]\epsilon\end{aligned}. \tag{12}$$

Since all the arm links have constant length, $\dot{T}_b = \vec{0}$, therefore

$$\begin{aligned}\hat{\xi}_f = \quad & \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b + \omega_b + [\omega_b \times T_b + \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b \times T_b \\ & + \boldsymbol{q}_b^* \otimes \omega_v \times (T_c + T_v) \otimes \boldsymbol{q}_b + \boldsymbol{q}_b^* \otimes \dot{T}_v \otimes \boldsymbol{q}_b]\epsilon\end{aligned}, \tag{13}$$

# Aerial manipulator dual quaternion modeling

## Robotic arm subsystem

Differentiating the twist of the main effector yields

$$\begin{aligned}\dot{\hat{\xi}}_f = & q_b^* \otimes \omega_v \otimes q_b \times \omega_b + q_b^* \otimes \dot{\omega}_v \otimes q_b + \dot{\omega}_b \\ & + \big[\dot{\omega}_b \times T_b + q_b^* \otimes \omega_v \otimes q_b \times \omega_b \times T_b \\ & + q_b^* \otimes \big(\dot{\omega}_v \times (T_c + T_v) + \omega_v \times \dot{T}_v\big) \otimes q_b \\ & + q_b^* \otimes \omega_v \times (T_c + T_v) \otimes q_b \times \omega_b \\ & + q_b^* \otimes \dot{T}_v \otimes q_b \times \omega_b + q_b^* \otimes \ddot{T}_v \otimes q_b \\ & + q_b^* \otimes \dot{\omega}_v \otimes q_b \times T_b\big]\epsilon\end{aligned} \quad . \tag{14}$$

The rotational and translational accelerations can be extracted from (14), resulting in the following expressions

$$\dot{\omega}_f \triangleq q_b^* \otimes \omega_v \otimes q_b \times \omega_b + q_b^* \otimes \dot{\omega}_v \otimes q_b + \dot{\omega}_b, \tag{15}$$

and

$$\ddot{T}_f \triangleq q_b^* \otimes \big(\ddot{T}_v + \dot{\omega}_v \times (T_c + T_v) + \omega_v \times \dot{T}_v\big) \otimes q_b. \tag{16}$$

# Aerial manipulator dual quaternion modeling
## Robotic arm subsystem

Finally, the dynamic model of the aerial manipulator's end effector is expressed as

$$\frac{d}{dt}\begin{bmatrix}\hat{q}_f \\ \hat{\xi}_f\end{bmatrix} = \begin{bmatrix} \begin{array}{l} \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b + \omega_b + \frac{1}{2}[-T_b \times \omega_b - T_b \times \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b \\ -\boldsymbol{q}_b^* \otimes (T_c + T_v) \times \omega_v \otimes \boldsymbol{q}_b + \boldsymbol{q}_b^* \otimes \omega_v \otimes \boldsymbol{q}_b \times T_b + \omega_b \times T_b \\ +\dot{T}_b + \boldsymbol{q}_b^* \otimes \omega_v \times (T_c + T_v) \otimes \boldsymbol{q}_b + 2\boldsymbol{q}_b^* \otimes \dot{T}_v \otimes \boldsymbol{q}_b]\epsilon \\ \\ \dot{\omega}_f + \left[\dot{\omega}_f \times T_b + q_b^* \otimes \left(\omega_v \times (T_c + T_v) + \dot{T}_v\right) \otimes q_b \times \omega_b + \ddot{T}_f\right]\epsilon. \quad , \end{array} \end{bmatrix} \tag{17}$$

# Aerial manipulator controller

## Dual Quaternion Chattering-free Sliding-mode Controller

Given two dual vectors $\hat{g} = \vec{g}_r + \vec{g}_d \epsilon$ and $\hat{h}_2 = \vec{h}_r + \vec{h}_d \epsilon$, let $\odot$ be a product between dual vectors defined as

$$\hat{g} \odot \hat{h} = \vec{g}_r \cdot \vec{h}_r + \vec{g}_d \cdot \vec{h}_d \epsilon, \tag{18}$$

note the output of product $\odot$ is a dual number with scalar components. Define a sliding surface with dual vectors as

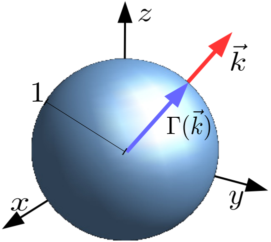
$$\hat{\gamma} = 2\hat{k}_a \cdot \ln \hat{q}_v + \hat{k}_b \cdot \hat{\xi}_v \rightarrow \dot{\hat{\gamma}} = \hat{k}_a \cdot \hat{\xi}_v + \hat{k}_b \cdot \hat{u}_v, \tag{19}$$

being $\hat{k}_a$ and $\hat{k}_b$ dual vectors containing positive constant entries.

# Aerial manipulator controller
## Dual Quaternion Chattering-free Sliding-mode Controller

Define a function $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$, which bounds any vector into a sphere respecting its original direction and modifying its magnitude in terms of an arctangent function as

$$\Gamma(\vec{\boldsymbol{k}}) = \hat{\boldsymbol{k}}\,\frac{2}{\pi}\tan^{-1}\left(||\vec{\boldsymbol{k}}||\right), \tag{20}$$



# Aerial manipulator controller

## Dual Quaternion Chattering-free Sliding-mode Controller

A dual scalar positive-definite function is then proposed as

$$\hat{V} = \frac{1}{2}\hat{\gamma} \odot \hat{\gamma} = V_r + V_d \epsilon, \quad V_r, V_d \in \mathbb{R}^+, \tag{21}$$

such that its derivative is given by

$$\dot{\hat{V}} = \hat{\gamma} \odot \dot{\hat{\gamma}} = \left(2\hat{k}_a \cdot \ln \hat{q}_v + \hat{k}_b \cdot \hat{\xi}_v\right) \odot \left(\hat{k}_a \cdot \hat{\xi}_v + \hat{k}_b \cdot \hat{u}_v\right), \tag{22}$$

expanding (22) yields

$$\begin{aligned} \dot{\hat{V}} &= \left(K_{ar}\vec{n}\vartheta + K_{br}\omega_v + [K_{ad}T_b + K_{bd}\dot{T}_v]\epsilon\right) \\ &\quad \odot \left(K_{ar}\omega_v + K_{br}J_v^{-1}\tau_v + [K_{ad}\dot{T}_v + m_v^{-1}K_{bd}F_v]\epsilon\right) \\ &= \left(K_{ar}\vec{n}\vartheta + K_{br}\omega_v\right) \cdot \left(K_{ar}\omega_v + K_{br}J_v^{-1}\tau_v\right) \\ &\quad + \left[(K_{ad}T_b + K_{bd}\dot{T}_v) \cdot (K_{ad}\dot{T}_v + m_v^{-1}K_{bd}F_v)\right]\epsilon \end{aligned}. \tag{23}$$

# Aerial manipulator controller

## Dual Quaternion Chattering-free Sliding-mode Controller

Proposing the components of $\hat{u}_v$ as

$$\tau_v = -J_v K_{br}^{-1}\left(K_{ar}\omega + K_{cr}\Gamma(K_{ar}\vec{n}\vartheta + K_{br}\omega)\right), \quad (24)$$

and

$$F_v = -mK_{bd}^{-1}\left(K_{ad}\dot{T}_v + K_{cd}\Gamma(K_{ad}T_v + K_{bd}\dot{T}_v)\right), \quad (25)$$

where $\hat{k}_c$ denotes a dual vector positive gain such that

$$\begin{aligned}\dot{\hat{V}} &= -\left(K_{ar}\vec{n}\vartheta + K_{br}\omega_v\right)\cdot K_{cr}\Gamma(K_{ar}\vec{n}\vartheta + K_{br}\omega) \\ &\quad -\left[(K_{ad}T_b + K_{bd}\dot{T}_v)\cdot K_{cd}\Gamma(K_{ad}T_v + K_{bd}\dot{T}_v)\right]\epsilon \; . \\ &= \dot{V}_r + \dot{V}_d\epsilon\end{aligned} \quad (26)$$

Note that $\dot{V}_r$ and $\dot{V}_d$ are negative definite scalar numbers. Since $V_r, V_d > 0 \; \forall \; \vec{n}\vartheta, \omega_v, T_v, \dot{T}_v \neq \bar{0}$ and $\dot{V}_r, \dot{V}_d < 0 \; \forall \; \vec{n}\vartheta, \omega_v, T_v, \dot{T}_v \neq \bar{0}$, then asymptotic stability is ensured for the quadrotor.

# Aerial manipulator controller
## Robotic arm dynamic effects

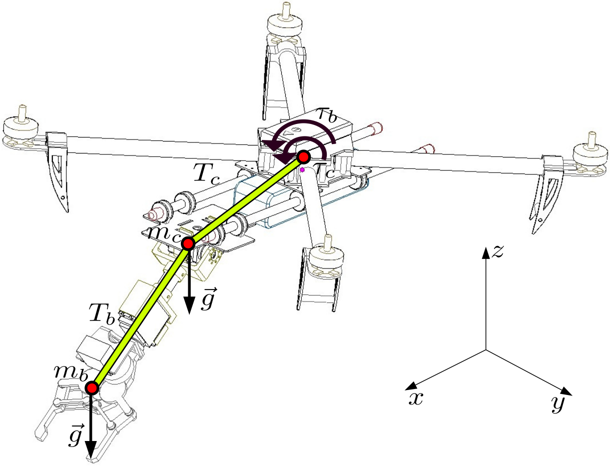



Figure : Aerial manipulator's free body diagram.

# Aerial manipulator controller

## Robotic arm dynamic effects

The total external torque $\tau_{arm}$ caused by the robotic limb can be defined by the sum of each individual link's torque

$$\tau_{arm} = \tau_c + \tau_b, \tag{27}$$

where $\tau_c$ defines the torque induced by the constant link $c$, and $\tau_b$ denotes the torque produced by the mobile link $b$, and

$$\tau_c = T_c \times m_c [q_v^* \otimes \bar{g} \otimes q_v] + \omega_v \times J_c \omega_v, \tag{28}$$

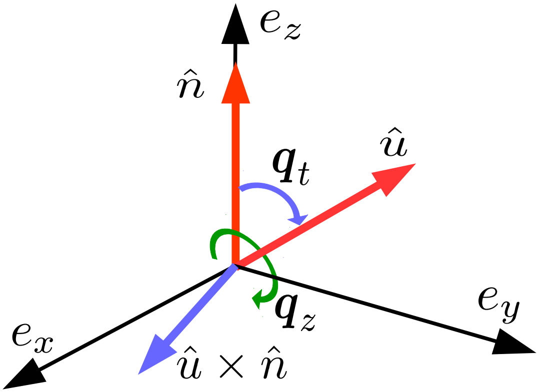
the link's mass is symbolized by $m_c$, $J_c$ represents its inertia matrix, and

$$\tau_b = (q_b \otimes T_b \otimes q_b^* + T_c) \times m_b q_v^* \otimes \bar{g} \otimes q_v + (\omega_v + \omega_b) \times J_b(\omega_v + \omega_b), \tag{29}$$

where $m_b$ denotes the mass of the link, and $J_b$ is its inertia matrix.

# Aerial manipulator controller

## Quadrotor Subactuation



# Aerial manipulator controller
## Quadrotor Subactuation

Defining a desired quaternion which rotates the vehicle's vertical trust towards the control force as

$$\boldsymbol{q_t} = \pm\left(\sqrt{\frac{1+\frac{F_v}{||F_v||}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}{2}} - \frac{\frac{F_v}{||F_v||}\times\begin{bmatrix}0\\0\\1\end{bmatrix}}{\left\|\frac{F_v}{||F_v||}\times\begin{bmatrix}0\\0\\1\end{bmatrix}\right\|}\sqrt{\frac{1-\frac{F_v}{||F_v||}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}{2}}\right), \quad (30)$$

If the desired rotation arround the $z$ axis is defined with a quaternion $\boldsymbol{q}_{zd}$, then $\boldsymbol{q}_t$ is completed using a quaternion product

$$\boldsymbol{q}_{vd} = \boldsymbol{q}_{zd} \otimes \boldsymbol{q}_t, \quad (31)$$

such that $q_{vd}$ is used as the complete attitude reference to feedback the rotational control

# Aerial manipulator controller
## Dual Quaternion Inverse Kinematics

Let the desired transformation of the end effector be defined as

$$\hat{q}_{fd} = q_{fd} + \frac{1}{2} q_{fd} \otimes T_{fd} \epsilon \tag{32}$$

where $T_{fd}$ and $q_{fd}$ denote the desired translation and rotation respectively.
The desired transformation $\hat{q}_{vr}$ where the quadrotor must converge can be computed by

$$\hat{q}_{vr} = \hat{q}_{fd} \otimes \hat{q}_b^* \otimes \hat{q}_c^*, \tag{33}$$

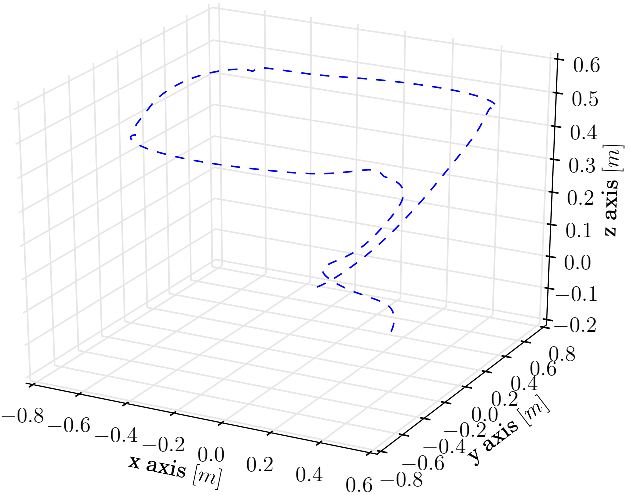
with

$$\hat{q}_{vr} = q_{vr} + \frac{1}{2} q_{vr} \otimes T_{vr} \epsilon, \tag{34}$$

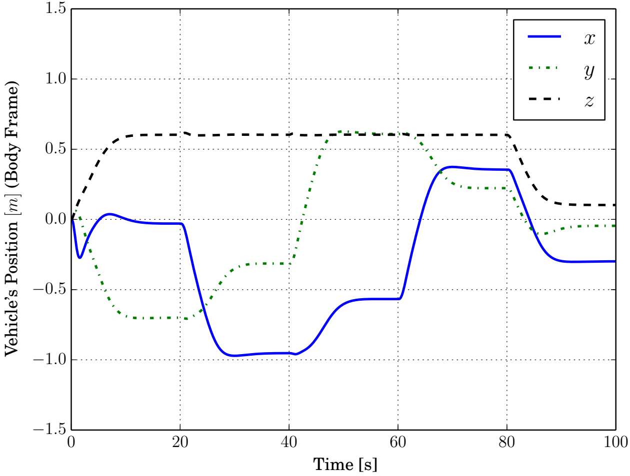
note $T_{vr}$ can be easily extracted from (34).
Since the desired attitude of the vehicle is defined by (31), then the total dual quaternion reference for the quadrotor can be defined as

$$\hat{q}_{vd} = q_{vd} + \frac{1}{2} q_{vd} \otimes T_{vr} \epsilon. \tag{35}$$

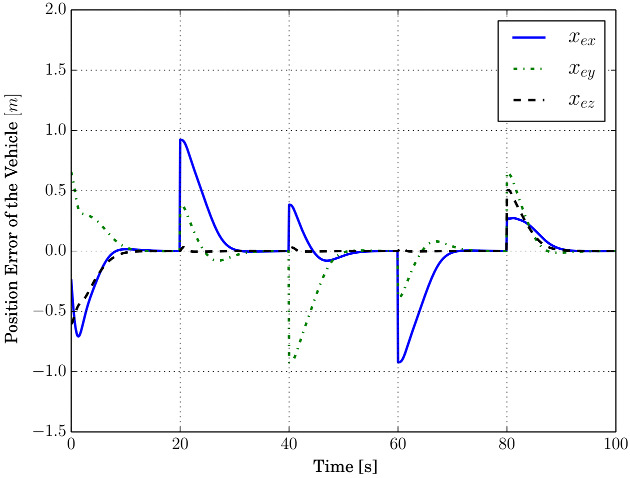
# Numerical Validation



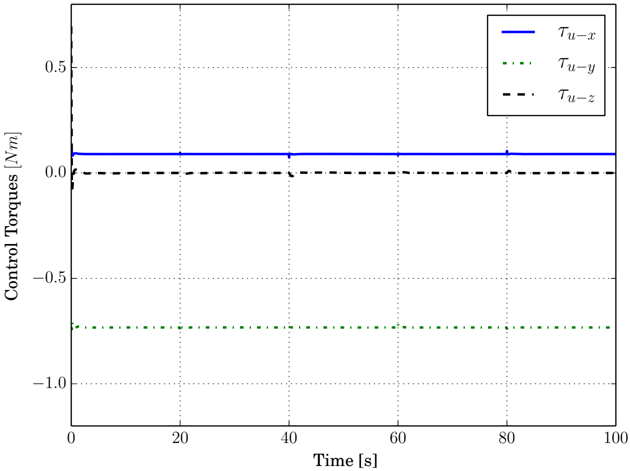
# Numerical Validation



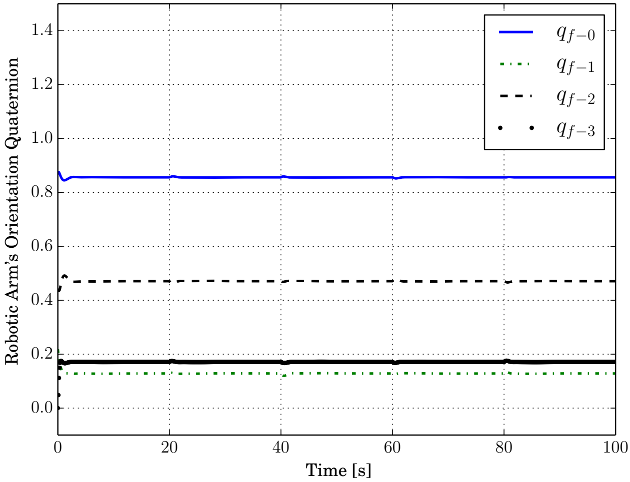
# Numerical Validation



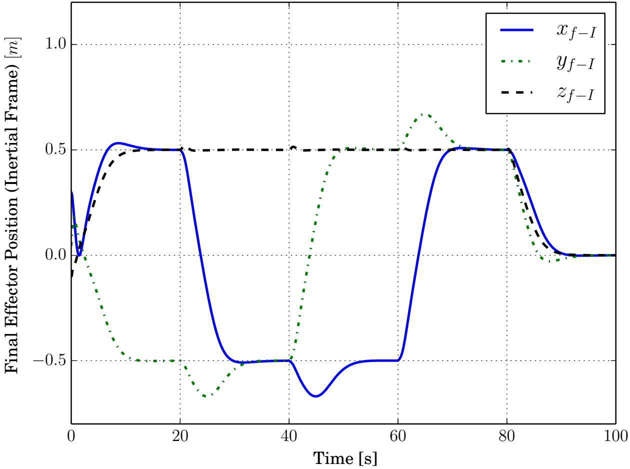
# Numerical Validation



# Numerical Validation



# Numerical Validation



# Conclusion

- Dual quaternions provide a shorter expression for complex transformations
- This approach can be easily expanded for more arm joints
- Dual quaternions and dual vectors simplify the development of controllers
- Future work:
  - Observers for torque uncertainties estimations
  - Experimental validations