GRADIENT-LIKE OBSERVERS FOR INVARIANT SYSTEMS

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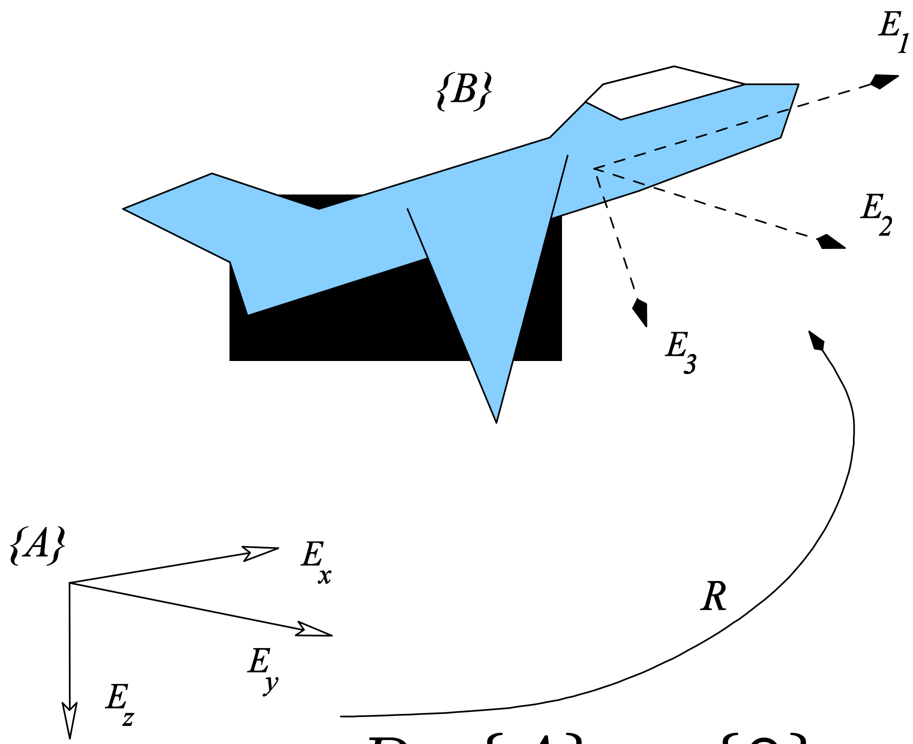
Motivation: Attitude estimation

- Most control algorithms depend on a good estimate of vehicle attitude.
- Many payload sensing systems depend explicitly on an accurate estimate of attitude.
- Gravitational direction is sufficient for some applications.



$$R : \{A\} \rightarrow \{0\}$$

Represents the attitude of the rigid-body airframe of the vehicle.

Attitude estimation on *SO*(3)

Attitude can be represented as: $R \in \mathbb{R}^{3\times 3}$ rotation matrices, or as $q \in \mathbb{R}^4$ unit quaternions, or as (ψ, θ, ϕ) roll-pitch-yaw Euler angles, or ....

Matricial representation:

$$R \in SO(3), \quad R^\top R = I, \qquad \Omega_\times \in \mathfrak{so}(3), \quad \Omega_\times^\top = -\Omega_\times$$

Left invariant kinematics

$$\dot{R} = R\Omega_\times$$

- $\Omega \in \{B\}$ measured in the body-fixed-frame.

Background

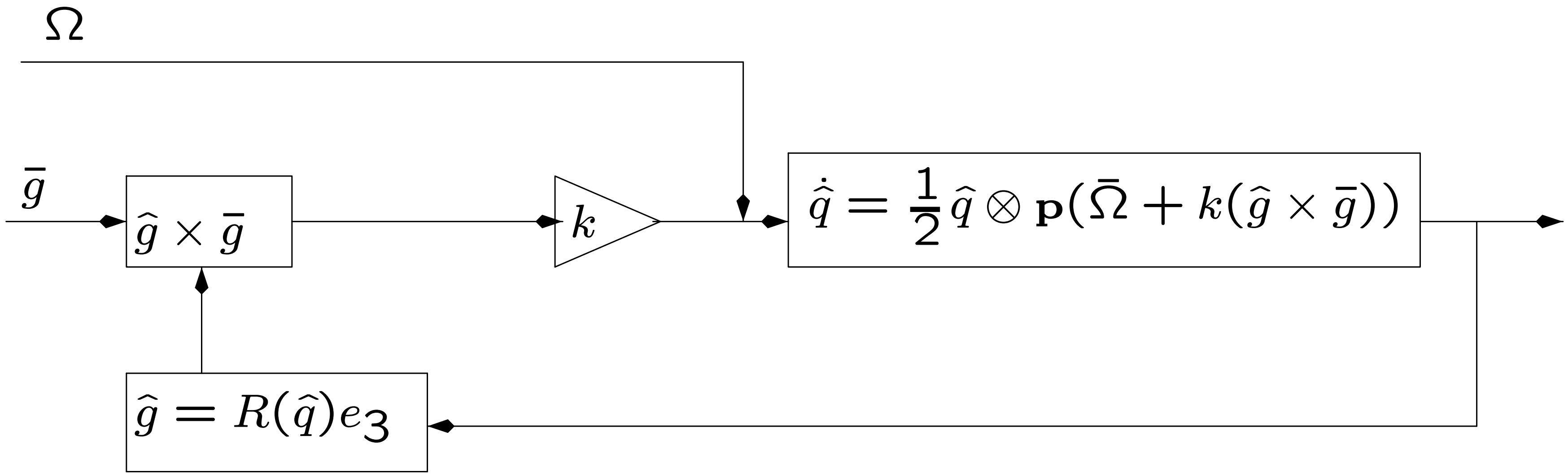
- Nonlinear observers for attitude estimation based on full state measurements have been available since Salcudean 1991.

Crassidis, J. L.; Markley, F. L. & Cheng, Y. "Nonlinear Attitude Filtering Methods", Journal of Guidance, Control and Dynamics, 2007, 30, 12-28

"Nonlinear observers are especially useful because they are often accompanied with global stability proofs. The property of guaranteed convergence from any initial condition is especially desired by designers of spacecraft attitude estimation applications. ***The observers in [47–52] all require an attitude measurement, which limits their use to cases where a point-by-point determined attitude is known.*** *Although nonlinear observers are still in their infancy, these methods show great promise for future applications."*

One of the most significant developments in the last 3 years is overcoming the requirement for point-by-point attitude measurements.

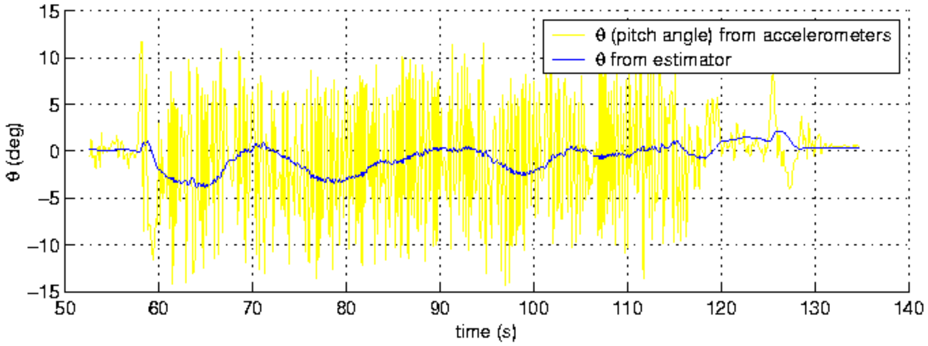
Explicit complementary filter

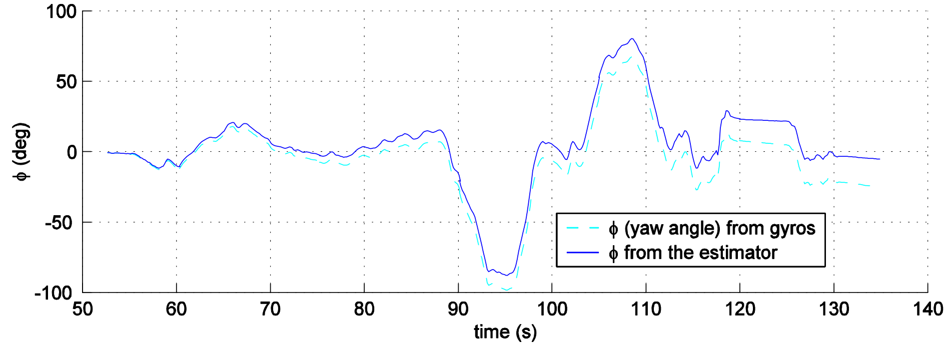


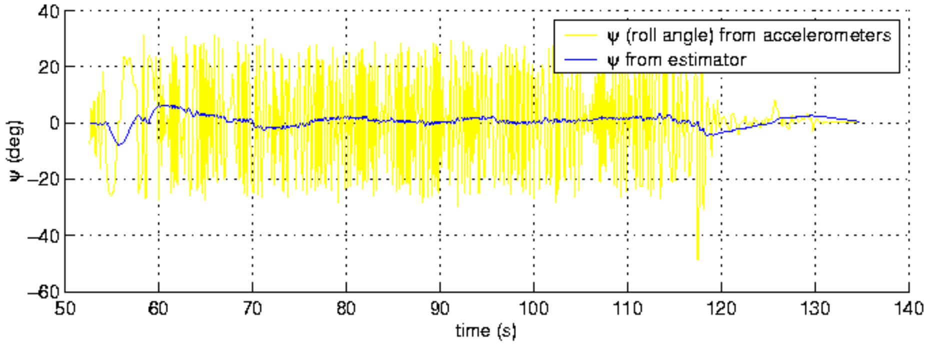
- Use quaternion representation for implementation.
- One update step is ~100flops @ 16 bit precision.
- Global stability properties common to NL-observers.
- Does not require reconstruction of attitude
  -the big downside of non-linear observers until 2006.

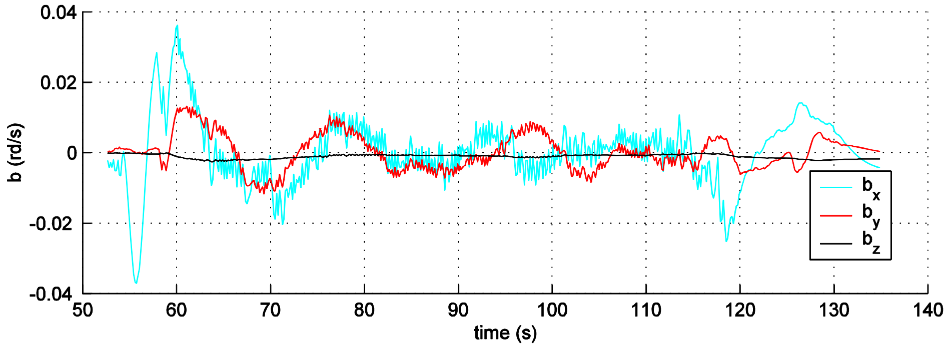
Hamel/Mahony 2006 (ICRA), Mahony/Hamel/Pflimlin 2008 (IEEE-TAC)

Results









- Implementation on several UAV systems are all highly successful.

- Easily add estimation of gyro bias.
- When accelerometers are used for attitude estimate this minimises drift in yaw estimate.

Motivation for general results

Strong interest in nonlinear observer design for invariant systems on Lie groups motivated by applications in robotics. eg:

- Attitude and pose tracking
  - Unmanned aerial, ground or submersible vehicles.
  - Virtual reality interfaces such as VR goggles, gloves.
- Homography tracking for vision.

Scientific interest in understanding the structure of systems with symmetry.

- Observability and controllability properties of such systems.
- Minimal realisations and structure theory.

Goal of this presentation

- Consider the general case of invariant dynamics acting on an arbitrary Lie group.
- Consider outputs generated by a group action.
- Provide a context for the development by deriving the explicit complementary filter for the attitude estimation problem.

Systems Model

System

$$\dot{X} = Xu, \quad u(t) \in \mathfrak{g}, \quad X(0) = X_0 \in G$$

Transitive group action acting **homogeneous output space** M

$$h : G \times M \to M, \qquad h : (X, y) \mapsto h(X, y)$$

Reference output $y_0 \in M$. Output

$$y := h(X, y_0)$$

Goal is to develop an observer

$$\dot{\hat{X}} = F(\hat{X}, y, y_0, u, t).$$

Matched outputs

A group action with the same invariance as the system is said to have **matched invariance**.

Left invariant system $\Rightarrow$ left invariant group action.

$$h(SX, u) = h(S, h(X, u))$$

Physically: Outputs are measured in the inertial frame

Attitude estimation example: eg. Partial pose extracted by computer vision algorithm.

- Known orientation $\bar{y}_0 \in S^2$ from model in body-fixed-frame

$$\bar{y} = R\bar{y}_0 \qquad \text{in inertial frame}$$

- Associated group action is

$$\bar{h} : G \times S^2 \to S^2, \quad h(R, \bar{y}_0) = R\bar{y}_0,$$

Complementary outputs

A group action with the opposite invariance as the system is said to have **complementary invariance**.

Left invariant system $\Rightarrow$ right invariant group action.

$$h(XS, u) = h(S, h(X, u))$$

Physically: Outputs are measured in the body-fixed-frame

Attitude estimation example: eg. magnetic vector

- Known orientation $y_0 \in S^2$ in inertial frame

$$y = R^\top y_0 \qquad \text{in body-fixed-frame}$$

- Associated group action is

$$h : G \times S^2 \to S^2, \quad h(R, y_0) = R^\top y_0,$$

Observability and symmetry

For complementary outputs the subgroup

$$\mathrm{stab}(y_0) := \{X \in G \,|\, h(X, y_0) = y_0\}$$

is a symmetry of the system.

System symmetry is the left invariance of the system

$$T_X L_S \dot{X} = S(Xu) = (SX)u$$

For complementary outputs this symmetry preserves the output

$$h(SX, y_0) = h(R_X S, y_0) = h(X, h(S, y_0)) = h(X, y_0)$$

The subgroup $\mathrm{stab}(y_0)$ is the unobservable subgroup in G.

Projected system

Consider an invariant system

$$\dot{X} = Xu$$

with complementary outputs $y = h(X, y_0)$

The **projected system** on the homogeneous output space M is defined by

$$\dot{y} = T_X h_{y_0}(Xu), \qquad X \in h_{y_0}^{-1}(y)$$

The projected system is well defined and provides a minimal observable realisation of the full system.

The projected system is not defined for matched outputs.

Attitude example

System

$$\dot{R} = R\Omega_\times, \qquad R \in SO(3), \quad \Omega_\times \in \mathfrak{so}(3)$$

Output for a known 'reference' $y_0 \in S^2$

$$y = h(R, y_0) = R^\top y_0$$

For arbitrary $R_y \in SO(3)$ such that $h(R_y, y_0) = R_y^\top y_0 = y$.

Projected system dynamics are

$$\dot{y} = T_X h_{y_0}(R_y \Omega_\times) = -\Omega_\times R_y^\top y_0 = -\Omega_\times y.$$

$$\dot{y} = -\Omega \times y$$

Projected system
defined on M

Proposed observer design methodology

The proposed approach is to develop observers for the projected system and then lift these to the Lie-group

1. Define an error to compare trajectories of systems M.

2. Define synchrony of trajectories.

3. Decompose observer into synchronous and innovation terms.

4. Define an equivariant cost function on M.

5. Choose the innovation equal to the gradient of the cost.

This defines an observer on the output space M

6. Define a lift from M up to the Lie group G.

7. Lift the observer to an observer on the Lie group.

Canonical Errors

Measure observer performance by an error

$$e : M \times M \to N,$$

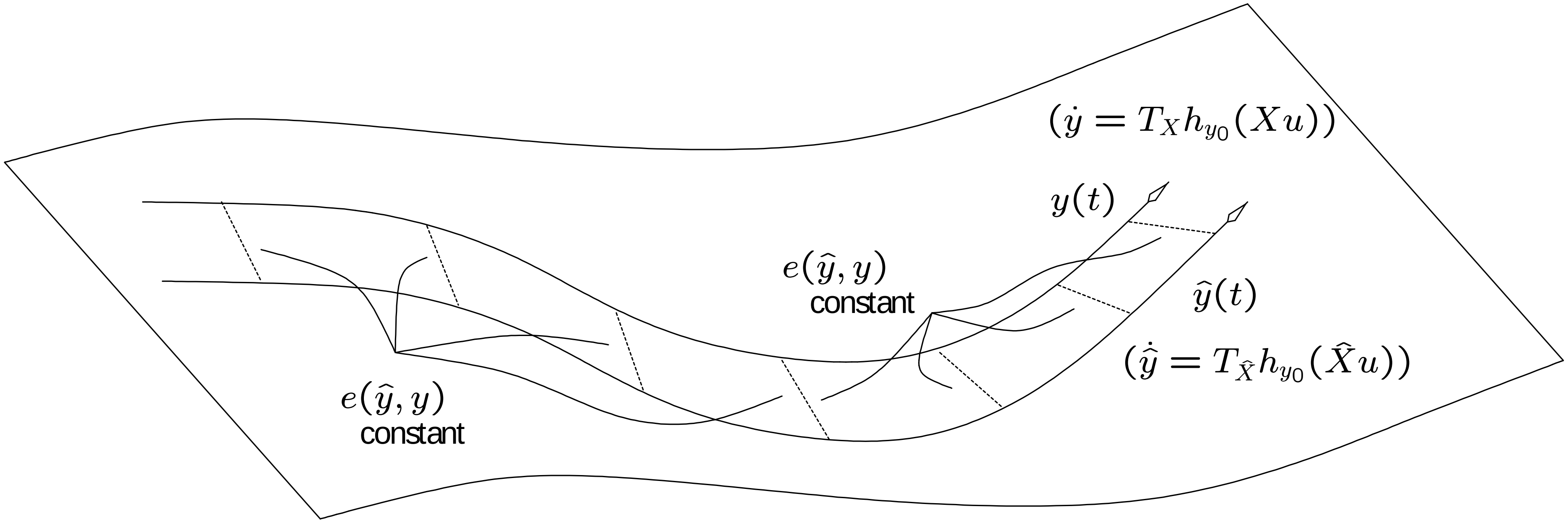
Two systems

$$\dot{y} = T_X h_{y_0}(Xu), \qquad \dot{\hat{y}} = F(\hat{y}, y, u, t)$$

are termed e **synchronous** if

$$\frac{d}{dt} e(\hat{y}(t), y(t)) = 0$$

Synchrony



Trajectories of two synchronous systems track each other.

A requirement for synchronisation of two systems is the measurement of the velocity *u*

Synchrony and invariant systems

Define a **canonical error**

$$\boxed{e(\hat{y}, y) = h(\hat{X}X^{-1}, y_0),} \qquad h(X, y_0) = y, \quad h(\hat{X}, y_0) = \hat{y}$$

- The pair of projected systems

$$\dot{y} = T_X h_{y_0}(Xu), \qquad \dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u)$$

are e synchronous

$$\frac{d}{dt} e(\hat{y}, y) = T_{\hat{X}X^{-1}} h_{y_0} \left(\frac{d}{dt} (\hat{X}X^{-1}) \right) = 0$$

- If these systems are $\bar{e}$ synchronous (for arbitrary $\bar{e}$) then

$$\bar{e}(\hat{y}, y) = g(e(\hat{y}, y)), \qquad g \colon M \to N$$

Canonical error

Synchrony and observer structure

Consider the pair of systems on M

$$\dot{y} = T_X h_{y_0}(Xu),$$
$$\dot{\hat{y}} = F(\hat{y}, y, u)$$

- If these systems are e synchronous then

$$F(\hat{y}, y, u) = T_{\hat{X}} h_{y_0}(\hat{X}u)$$

Thus, for invariant systems with complementary outputs, the natural choice of synchronous coupled system is

$$\dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u)$$

and the correct concept of error is the cannonical error.

Decomposition of observers

Consider a system and an observer on G

$$\dot{y} = T_X h_{y_0}(Xu), \qquad \dot{\hat{y}} = \hat{F}(\hat{y}, y, u)$$

If the observer has an *internal model* of the system (identical initial conditions lead to identical trajectories) then $\hat{F}(\hat{y}, y, u)$ can be decomposed into

$$\boxed{\hat{F}(\hat{y}, y, u) = \underbrace{T_{\hat{X}} h_{y_0}(\hat{X}u)}_{\text{synchronous}} + \underbrace{\alpha(\hat{y}, y, u)}_{\text{innovation}}}$$

where an innovation has the property that $\alpha(\hat{y}, y, u) = 0$.

Synchronous term induces no dynamics in e. The observer error dynamics are driven purely by the innovation term α.

Autonomous error dynamics

System and observer

$$\dot{y} = T_X h_{y_0}(Xu), \qquad \dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u) + \alpha(\hat{y}, y, u, t)$$

The dynamics of the canonical error of this observer is **autonomous** if and only if

- $\alpha(\hat{y}, y, u, t) = \alpha(\hat{y}, y)$ is independent of u and t.
- $T_{\hat{y}} h_X \alpha(\hat{y}, y, u, t) = \alpha(h_X(\hat{y}), h_X(y), u, t)$ is equivariant

The error dynamics of the right canonical error are

$$\dot{e} = \alpha(e(\hat{y}, y), y_0)$$

Perspectives

There are a number of important consequences of the property of autonomous error dynamics.

$$\dot{e} = \alpha(e(\hat{y}, y), y_0)$$

• Linearisation of the error dynamics is constant along trajectories. Use linear design concepts to design α.

• Analyse stability of e globally on M using comparison functions.

• Design α as a gradient like descent term.

Let $\widehat{f} : M \to \mathbb{R}$ be a Morse-Bott function with a unique global minima at y_0. Define an equivariant cost

$$f(y_1, y_2) = f(h(X, y_0), h(\hat{X}, y_0)) = \widehat{f}(h(\hat{X}^{-1}X, y_0)).$$

- A **reductive** homogeneous admits an invariant "normal" metric

$$\langle u, v \rangle = \langle T_y h_X u, T_y h_X v \rangle.$$

- The gradient of f with respect to $\widehat{y}$ is equivariant

$$T_{\widehat{y}} h_X \, \mathrm{grad}_1 \, f(\widehat{y}, y) = \mathrm{grad}_1 \, f(h_X(\widehat{y}), h_X(y)).$$

Observers for projected system

We propose the observer design

$$\boxed{\dot{\hat{y}} = T_{\widehat{X}} h_{y_0}(\widehat{X} u) - \operatorname{grad}_1 f(\hat{y}, y)}$$

The error e converges to y_0 for almost all initial conditions and arbitrary, admissible inputs u.

The error dynamics obtained are

$$\frac{d}{dt} e = -\operatorname{grad}_1 f(e, y_0).$$

One has

$$\frac{d}{dt} f(e, y_0) = -\|\operatorname{grad}_1 f(e, y_0)\|^2$$

Leads to strong stability results as expected.

Stability result

Consider the system/observer pair

$$\dot{y} = T_X h_{y_0}(Xu), \qquad \dot{\hat{y}} = T_{\hat{X}} h_{y_0}(\hat{X}u) - \mathrm{grad}_1\, f(\hat{X}, X)$$

where f is a right invariant cost function derived from a Morse-Bott cost $\tilde{f}$, and grad_1 is computed with respect to the normal metric.

Then $e(\hat{y}(t), y(t))$ converges almost globally asymptotically and locally exponentially to y_0. It follows that $\hat{y}(t) \to y(t)$ almost globally and locally exponentially.

Example: Attitude filter on SO(3)

Projected system

$$\dot{y} = -\Omega_\times y$$

Cost on S^2 derived from $\hat{f}(y) = 1 - y^\top y_0$,

$$f(\hat{y}, y) = k(1 - \hat{y}^\top y)$$

Normal metric on S^2 is just $\langle v, w\rangle = v^\top w$ for $v, w \in T_y S^2$

Gradient of f

$$\langle \mathrm{grad}_{\hat{y}} f, w\rangle = D_{\hat{y}} f[w] = -k w^\top y = w^\top \underbrace{\left(-k\left(I - \hat{y}\hat{y}^\top\right) y\right)}_{\mathrm{grad}_{\hat{y}} f}$$

Proposed observer

$$\dot{\hat{y}} = -\Omega_\times \hat{y} + k\left(I - \hat{y}\hat{y}^\top\right) y$$

Attitude observers

1: Note that

$$k(y\hat{y}^\top - \hat{y}y^\top)\hat{y} = k(y - \hat{y}y^\top\hat{y}) = k\left(I - \hat{y}\hat{y}^\top\right)y = -\,\mathrm{grad}_1 f(\hat{y}, y)$$

2: Moreover, one has

$$(\hat{y}y^\top - y\hat{y}^\top) = (\hat{y} \times y)_\times$$

Proposed observer

$$\dot{\hat{y}} = -\,\Omega_\times \hat{y} + k(\hat{y} \times y)_\times \hat{y}$$

Metni/Pflimlin/Hamel/Soueres : IROS, 2005
Metni/Pflimlin/Hamel/Soueres : *CEP,* 2006.

These filters can be interpreted as the natural gradient-like observer for the projected system.

Lift to the Lie group

For M reductive there exists a horizontal distribution $H(X)$ and invariant metric g on G such that the projection of g from $H(X)$ onto TM induces the invariant Riemannian metric on M.

Given $H(X)$ there is a unique linear map, termed the lift,

$$(\cdot)^H : T_y M \to H(X)$$

such that $T_X h_{y_0}((v)^H) = v$.

Lift the observer on M up to G

$$\dot{\hat{X}} = \hat{X}u - \Big(\mathrm{grad}_1\, f(h_{y_0}(\hat{X}), y)\Big)^H$$

Lift for SO(3) acting on S²

The horizontal distribution is

$$H(\hat{R}) = \{\omega_\times \hat{R} \in T_{\hat{R}} SO(3) \,\big|\, \omega^\top y_0 = 0\}$$

Consider a vector $v \in T_{\hat{y}} S^2$, $\hat{y} = \hat{R}^\top y_0$.
Define $\bar{\Omega}(v)$ by the solution to

$$\bar{\Omega}(v) \times \hat{y} = v, \quad \bar{\Omega}(v)^\top \hat{y} = 0.$$

The horizontal lift $(v)^H$ is defined by

$$(v)^H := \left(\mathrm{Ad}_{\hat{R}} \bar{\Omega}_\times\right) \hat{R} = \hat{R}\bar{\Omega}_\times.$$

Can be verified that the right invariant metric projects to the standard metric on S^2.

Computing the lift to *SO*(3)

The observer on the Lie group is given by the lift of the output observer

$$\dot{\hat{R}} = \hat{R}\Omega_\times + k\left((I - \hat{y}\hat{y}^\top)y\right)^H$$

Recall that

$$\mathrm{grad}_1 f(\hat{y}, y) = k(I - \hat{y}\hat{y}^\top)y = k(\hat{y} \times y) \times \hat{y}$$

Thus, by definition

$$\bar{\Omega}(\mathrm{grad}_1 f) \times \hat{y} = k(\hat{y} \times y) \times \hat{y} \Rightarrow \bar{\Omega}(\mathrm{grad}_1 f) = k(\hat{y} \times y)$$

Partial attitude filter

The proposed observer on $SO(3)$ is

$$\boxed{\dot{\hat{R}} = \hat{R}\left(\Omega_\times + k(\hat{y} \times y)_\times\right), \quad \hat{R}(0).}$$

Set $y = \bar{g}$ measure of graviaty and $\hat{g} = \hat{R}^\top e_3$

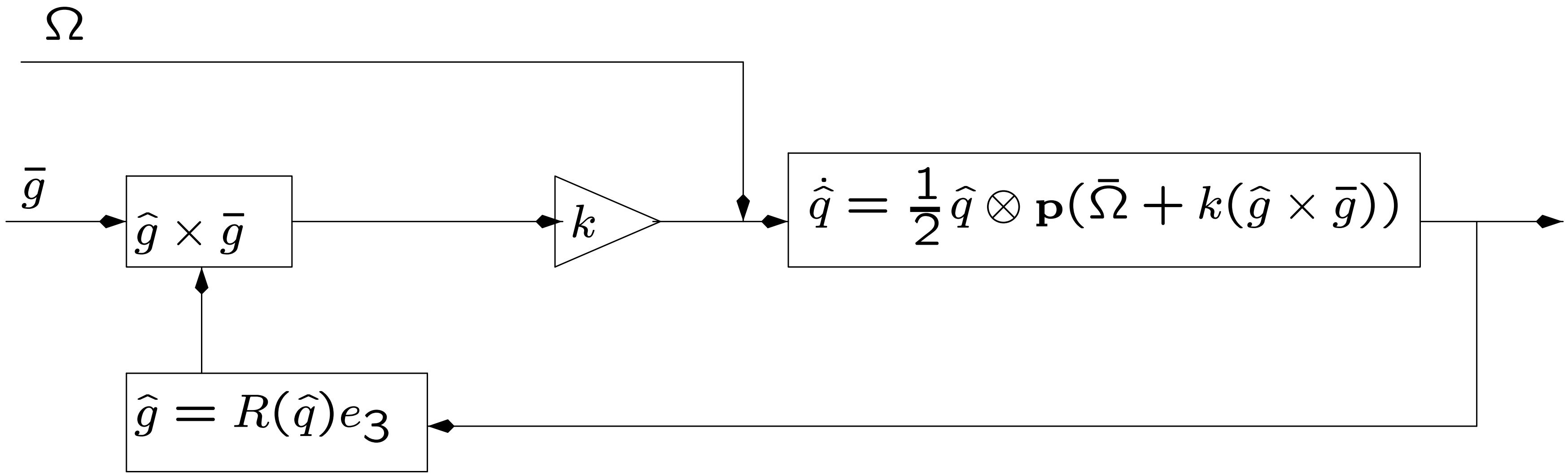
$$\dot{\hat{R}} = \hat{R}\left(\Omega_\times + k(\hat{g} \times g)_\times\right), \quad \hat{R}(0).$$

This is the explicit complementary filter discussed in Mahony/Hamel/Pflimlin Trans. Automatic Control 2008

In quaternion representation one has

$$\boxed{\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes \mathbf{p}(\Omega_y + 2k(\hat{g} \times \bar{g}))}$$

Explicit complementary filter



The explicit complementary filter is the natural lift onto $SO(3)$ of a gradient-like observer for an inertial direction on S^2.

All the early observer designs with global stability were designed by “guessing” a good form for the innovation.

Applications

- **Attitude estimation**
  - All UAV drones. Sensor packages.
- **Homography tracking** for vision sequences of planar surfaces.
  - High altitude observation of the ground.
- **Attitude heading reference systems (AHRS)**
  - Fixed wing UAV drones.
- **Pose reconstruction** from point measurements.
  - Indoors applications with LIDAR or similar systems.
- **Inertial vision pose reconstruction** from point feature measurements.
  - VTOL UAV systems operating in cluttered environments. Indoors, Urban canyons, etc.

Conclusions

- Symmetry and structure can be exploited effectively in the design of observers for invariant systems.
- Systems with complementary outputs form a natural physical model of many real world systems.
- Systems with complementary outputs lead to a projected system on the output homogeneous space with natural structure.
- Given a sensible cost function the resulting gradient-like observers are an alternative to complex nonlinear filter design.

Collaborators

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